

Product Formula, Independence and Asymptotic Moment-Independence for Complex Multiple Wiener-Itô Integrals

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Abstract We present a product formula for complex multiple Wiener-Itô integrals. As applications, we show Üstünel-Zakai independence criterion and the Nourdin-Rosiński asymptotic moment-independent criterion for complex multiple Wiener-Itô integrals.

Keywords: Complex Multiple Wiener-Itô Integrals; Product Formula; Üstünel-Zakai independence criterion; Asymptotic Moment-Independent.

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1 Introduction

The product formula of real multiple Wiener-Itô integrals is well known. Using this formula there are many interesting findings such as Üstünel-Zakai independence criterion [8, 16] for two multiple Wiener-Itô integrals, Nourdin-Rosiński asymptotic moment-independence criterion between blocks consisting of multiple Wiener-Itô integrals [10] and Fourth Moment Theorem (or say: Nualart-Peccati criterion) of a normalized sequence of real multiple Wiener-Itô integrals in a fixed Wiener chaos [9, 12].

Both real multiple Wiener-Itô integrals and complex multiple Wiener-Itô integrals (see [7] or Section 2 below) were established by Itô K. almost at the same time [6, 7]. However, the product formula of complex multiple Wiener-Itô integrals is still unknown up to now as far as we know. The key aim of this paper is to answer this question (see Theorem 3.2). As far as we know, there exist at least three different approaches to prove the product formula. In this paper, we adopt the most simple one by using the relationship between complex multiple Wiener-Itô integrals and complex Hermite polynomials given by Itô [7].

As applications, we will show Üstünel-Zakai independence criterion, i.e., a necessary and sufficient condition on the pair of kernels (f, g) is derived under which

the complex multiple Wiener-Itô integrals $I_{a,b}(f)$, $I_{c,d}(g)$ are independent (see Theorem 4.2). In the final, by using the connection between real Wiener-Itô integrals and complex Wiener-Itô integrals [3, Theorem 3.3], we list two related results as an appendix: Nourdin-Rosiński asymptotic moment-independence criterion and joint convergence criterion for d -dimensional vectors consisting of complex multiple Wiener-Itô integrals (see Theorem 5.2, Corollary 5.3).

2 Preliminaries

In this section, we shortly recall the theory of complex multiple Wiener-Itô integrals of Itô [7]. Consider a triple (T, \mathcal{B}, μ) , where the measure μ is positive, σ -finite and non-atomic. $\mathfrak{H} = L^2(T, \mathcal{B}, \mu)$ is a complex separable Hilbert space. A complex Gaussian random measure over (T, \mathcal{B}) , with control μ , is a centered complex Gaussian family of the type

$$\mathbf{M} = \{M(B) : B \in \mathcal{B}, \mu(B) < \infty\},$$

such that, for every $B, C \in \mathcal{B}$ with finite measure,

$$E[M(B)\overline{M(C)}] = \mu(B \cap C).$$

Notation 1. For a fixed (p, q) , suppose that $f \in \mathfrak{H}^{\otimes(p+q)}$. \hat{f} is the symmetrization of f in the sense of Itô [7]:

$$\tilde{f}(t_1, \dots, t_{p+q}) = \frac{1}{p!q!} \sum_{\pi} \sum_{\sigma} f(t_{\pi(1)}, \dots, t_{\pi(p)}, t_{\sigma(1)}, \dots, t_{\sigma(q)}), \quad (2.1)$$

where π and σ run over all permutations of $(1, \dots, p)$ and $(p+1, \dots, p+q)$ respectively. Denote by $\mathfrak{H}^{\odot p} \otimes \mathfrak{H}^{\odot q} = L_S^2(T^p, \mathcal{B}^{\otimes p}, \mu^{\otimes p}) \otimes L_S^2(T^q, \mathcal{B}^{\otimes q}, \mu^{\otimes q})$ the space of square integrable and symmetric functions on T^{p+q} in the above sense. Notice that (2.1) is different to the ordinary symmetrization of f in the theory of real multiple integrals which is given by

$$\hat{f}(t_1, \dots, t_{p+q}) = \frac{1}{(p+q)!} \sum_{\pi} f(t_{\pi(1)}, \dots, t_{\pi(p+q)}), \quad (2.2)$$

where π runs over all permutations of $(1, \dots, p+q)$.

Obviously, we have that (see (5.2) of [7])

$$\|\tilde{f}\| \leq \|f\|. \quad (2.3)$$

Definition 2.1. (Complex multiple Wiener-Itô integrals [7]) Suppose that $E_1, \dots, E_n \subset \mathcal{B}$ is any system of disjoint sets and $e_{i_1 \dots i_p j_1 \dots j_q}$ is a complex-valued function defined for $i_1, \dots, i_p, j_1, \dots, j_q = 1, 2, \dots, n$ such that $e_{i_1 \dots i_p j_1 \dots j_q} = 0$ unless $i_1, \dots, i_p, j_1, \dots, j_q$ are all different. Let \mathcal{S}_{pq} denote the set of all functions of the form

$$f(t_1, \dots, t_p, s_1, \dots, s_q) = \sum e_{i_1 \dots i_p j_1 \dots j_q} \mathbf{1}_{E_{i_1} \times \dots \times E_{i_p} \times E_{j_1} \times \dots \times E_{j_q}}, \quad (2.4)$$

where $\mathbf{1}_B(\cdot)$ is the characteristic function of the set B . The multiple integral of f is defined by

$$I_{p,q}(f) = \sum e_{i_1 \dots i_p j_1 \dots j_q} M(E_{i_1}) \dots M(E_{i_p}) \overline{M(E_{j_1})} \dots \overline{M(E_{j_q})}. \quad (2.5)$$

Clearly, the above integral satisfies that

$$I_{p,q}(f) = I_{p,q}(\tilde{f}), \quad (2.6)$$

$$E[I_{p,q}(f) \overline{I_{p,q}(g)}] = p!q! \langle \tilde{f}, \tilde{g} \rangle, \quad (2.7)$$

$$E[|I_{p,q}(f)|^2] = p!q! \|\tilde{f}\|^2 \leq p!q! \|f\|^2, \quad (\text{Itô's isometry}) \quad (2.8)$$

where $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$ are norm and inner product on $\mathfrak{H}^{\otimes(p+q)}$. Since \mathcal{S}_{pq} is dense in $\mathfrak{H}^{\otimes(p+q)}$, one can extend the integral to any $f \in \mathfrak{H}^{\otimes(p+q)}$ by taking the limit, i.e.,

$$I_{p,q}(f) := \int \dots \int f dM(t_1) \dots dM(t_p) \overline{dM(s_1)} \dots \overline{dM(s_q)} = \lim_n I_{p,q}(f_n), \quad (2.9)$$

where $f_n \in \mathcal{S}_{pq}$ such that $f_n \rightarrow f$ in $\mathfrak{H}^{\otimes(p+q)}$, and the definition is independent of the choice of the sequence $\{f_n\}$. In addition, (2.6-2.8) are still valid to any $f, g \in \mathfrak{H}^{\otimes(p+q)}$. Moreover, the set

$$\mathcal{H}_{p,q} := \{I_{p,q}(f) : f \in \mathfrak{H}^{\otimes(p+q)}\}$$

is called the Wiener-Itô chaos of degree of (p, q) or (p, q) -th Wiener-Itô chaos.

Definition 2.2. (Complex Hermite polynomials) The complex Hermite polynomials $J_{m,n}(z, \rho)$ are given by [7]

$$\exp \{ \lambda \bar{z} + \bar{\lambda} z - \rho |\lambda|^2 \} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\bar{\lambda}^m \lambda^n}{m!n!} J_{m,n}(z, \rho), \quad (2.10)$$

where $\lambda \in \mathbb{C}$. When $\rho = 2$, we will often write $J_{m,n}(z)$ rather than $J_{m,n}(z, \rho)$.

Applying [7, Theorem 9] (or see Lemma 3.1 below) and the properties of complex Hermite polynomials (see [7, Theorem 12]), Itô established the relation between complex multiple integrals and complex Hermite polynomials [7, Theorem 13.2]: suppose that $h_1(t), \dots, h_l(t)$ be any orthonormal system in \mathfrak{H} and $\alpha_i, \beta_j = 1, \dots, l$, then

$$\begin{aligned} & \int \dots \int h_{\alpha_1}(t_1) \dots h_{\alpha_m}(t_m) \overline{h_{\beta_1}(s_1)} \dots \overline{h_{\beta_n}(s_n)} dM(t_1) \dots dM(t_m) \overline{dM(s_1)} \dots \overline{dM(s_n)} \\ &= \prod_{k=1}^l 2^{-\frac{m_k + n_k}{2}} J_{m_k, n_k}(\sqrt{2} Z(h_k)), \end{aligned} \quad (2.11)$$

where

$$Z(h_k) = \int h_k(t) dM(t), \quad k = 1, \dots, l, \quad (2.12)$$

and m_k, n_k are the number of k appeared in α_i and β_j respectively.

Remark 1. As a result of the above equality and Proposition 2.9 of [3], $\mathcal{H}_{p,q}$ is equal to the closed linear subspace of $L^2_{\mathbb{C}}(\mathbf{M})$ generated by the random variables of the type

$$\left\{ J_{m,n}(Z(h)), h \in \mathfrak{H}, \|h\|_{\mathfrak{H}} = \sqrt{2} \right\}, \quad (2.13)$$

where $Z(h)$ is the same as (2.12). Please refer to Definition 2.7 and Remark 9 of [3] for details.

Notation 2. Suppose $f(t_1, \dots, t_p, s_1, \dots, s_q) \in \mathfrak{H}^{\odot p} \otimes \mathfrak{H}^{\odot q}$. We call

$$\mathfrak{H}^{\odot q} \otimes \mathfrak{H}^{\odot p} \ni h(t_1, \dots, t_q, s_1, \dots, s_p) := \bar{f}(s_1, \dots, s_p, t_1, \dots, t_q)$$

the **reversed complex conjugate** of function $f(t_1, \dots, t_p, s_1, \dots, s_q)$.

From Definition 2.1, we can obtain the following lemma easily.

Lemma 2.3. Suppose $f(t_1, \dots, t_p, s_1, \dots, s_q) \in \mathfrak{H}^{\odot p} \otimes \mathfrak{H}^{\odot q}$. Let h be the reversed complex conjugate of f , then

$$\overline{I_{p,q}(f)} = I_{q,p}(h). \quad (2.14)$$

We conclude these preliminaries by two propositions, that will be needed throughout the sequel.

Proposition 2.4. (1) *Complex multiple Wiener-Itô integrals have all moments satisfying the following hypercontractivity inequality*

$$[E |I_{p,q}(f)|^r]^{\frac{1}{r}} \leq (r-1)^{\frac{p+q}{2}} [E |I_{p,q}(f)|^2]^{\frac{1}{2}}, \quad r \geq 2, \quad (2.15)$$

where $|\cdot|$ is the absolute value (or modulus) of a complex number.

(2) *If a sequence of distributions of $\{I_{p,q}(f_n)\}_{n \geq 1}$ is tight, then*

$$\sup_n E |I_{p,q}(f)|^r < \infty \quad \text{for every } r > 0. \quad (2.16)$$

Proof. (i) (2.15) is a consequence of the hypercontractivity of normal Ornstein-Uhlenbeck semigroup [1].

(ii) Along the same line as (ii) of [10, Lemma 2.1] for the case of real multiple integrals, we can show that (2.16) holds. \square

Set $\mathbf{M} = \frac{1}{\sqrt{2}}[\mathbf{M}_1 + i\mathbf{M}_2]$, $\mathbf{M}_1, \mathbf{M}_2$ are two real independent continuous normal system. Let $\hat{T} = \{1, 2\} \times T$, $\mathcal{B}(\hat{T}) = \mathcal{B}(\{1, 2\} \times T)$,

$$\widehat{M}(B) = M_1(B_1) + M_2(B_2), \quad \forall B = (\{1\} \times B_1) \cup (\{2\} \times B_2) \in \mathcal{B}(\hat{T}).$$

Then $L^2(\hat{T}) = L^2(T) \oplus L^2(T)$ and

$$\widehat{\mathbf{M}} = \left\{ \widehat{M}(B) : B = (\{1\} \times B_1) \cup (\{2\} \times B_2), \mu(B_1) + \mu(B_2) < \infty \right\}$$

is a real normal random measure on $(\hat{T}, \mathcal{B}(\hat{T}))$. Denote by $I_n(f)$ the real n -th multiple Wiener-Itô integral of f with respect to $\widehat{\mathbf{M}}$ (see subsection 3.2 of [3]).

Proposition 2.5. *Suppose that $h \in \mathfrak{H}^{\otimes p} \otimes \mathfrak{H}^{\otimes q}$. Then there exist $f, g \in (L^2(\widehat{T}))^{\otimes(p+q)}$ such that*

$$I_{p,q}(h) = I_{p+q}(f) + i I_{p+q}(g). \quad (2.17)$$

That is, both of the real part and the imaginary part of a complex multiple integral can be represented by real multiple integrals [3, Theorem 3.3].

3 The product formula for complex multiple Wiener-Itô integrals

Notation 3. For two symmetric functions $f \in \mathfrak{H}^{\odot p_1} \otimes \mathfrak{H}^{\odot q_1}$, $g \in \mathfrak{H}^{\odot p_2} \otimes \mathfrak{H}^{\odot q_2}$ and $i \leq p_1 \wedge q_2$, $j \leq q_1 \wedge p_2$, the contraction of (i, j) indices of the two functions is given by

$$\begin{aligned} & f \otimes_{i,j} g(t_1, \dots, t_{p_1+p_2-i-j}; s_1, \dots, s_{q_1+q_2-i-j}) \\ &= \int_{A^{i+j}} \mu^{i+j}(du_1 \cdots du_i dv_1 \cdots dv_j) f(t_1, \dots, t_{p_1-i}, u_1, \dots, u_i; s_1, \dots, s_{q_1-j}, v_1, \dots, v_j) \\ & \times g(t_{p_1-i+1}, \dots, t_{p_1-i+p_2-j}, v_1, \dots, v_j; s_{q_1-j+1}, \dots, s_{q_1-j+q_2-i}, u_1, \dots, u_i); \end{aligned} \quad (3.18)$$

by convention, $f \otimes_{0,0} g = f \otimes g$ denotes the tensor product of f and g . We write $f \tilde{\otimes}_{p,q} g$ for the symmetrization of $f \otimes_{p,q} g$. In what follows, we use the convention $f \otimes_{i,j} g = 0$ if $i > p_1 \wedge q_2$ or $j > q_1 \wedge p_2$.

The following lemma is the starting point of the relationship (2.11) and the product formula for complex multiple Wiener-Itô integrals.

Lemma 3.1. [7, Theorem 9] *Let $f \in \mathfrak{H}^{\odot p} \otimes \mathfrak{H}^{\odot q}$ be a symmetric function and let $g \in \mathfrak{H}$. Then*

$$I_{p,q}(f)I_{1,0}(g) = I_{p+1,q}(f \otimes g) + qI_{p,q-1}(f \otimes_{0,1} g), \quad (3.19)$$

$$I_{p,q}(f)I_{0,1}(g) = I_{p,q+1}(f \otimes g) + pI_{p-1,q}(f \otimes_{1,0} g). \quad (3.20)$$

Theorem 3.2. (Product formula) *For two symmetric functions $f \in \mathfrak{H}^{\odot a} \otimes \mathfrak{H}^{\odot b}$, $g \in \mathfrak{H}^{\odot c} \otimes \mathfrak{H}^{\odot d}$, the product formula for complex multiple Wiener-Itô integrals is given by*

$$I_{a,b}(f)I_{c,d}(g) = \sum_{i=0}^{a \wedge d} \sum_{j=0}^{b \wedge c} \binom{a}{i} \binom{d}{i} \binom{b}{j} \binom{c}{j} i!j! I_{a+c-i-j, b+d-i-j}(f \otimes_{i,j} g). \quad (3.21)$$

where $a, b, c, d \in \mathbb{N}$.

Proof. From Remark 1, we only need to show (3.21) hold for $I_{a,b}(f) = J_{a,b}(Z(h_1))$ and $I_{c,d}(g) = J_{c,d}(Z(h_2))$ with $h_1, h_2 \in \mathfrak{H}$ such that $\|h_1\| = \|h_2\| = \sqrt{2}$ by a density argument. That is to say, $f = h_1^{\otimes a} \otimes \bar{h}_1^{\otimes b}$, $g = h_2^{\otimes c} \otimes \bar{h}_2^{\otimes d}$. By the decomposition theorem of Hilbert spaces[14, p71], we may as well assume that $h_1 = h_2$ or $\langle h_1, h_2 \rangle_{\mathfrak{H}} = 0$.

It follows from the generating function of complex Hermite polynomials [2, 7] that

$$\begin{aligned}
& \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} \frac{\bar{\lambda}^a \lambda^b}{a!b!} J_{a,b}(z) \sum_{c=0}^{\infty} \sum_{d=0}^{\infty} \frac{\bar{\mu}^c \mu^d}{c!d!} J_{c,d}(z) \\
&= \exp \{ \lambda \bar{z} + \bar{\lambda} z - 2|\lambda|^2 \} \exp \{ \mu \bar{z} + \bar{\mu} z - 2|\mu|^2 \} \\
&= \exp \left\{ (\lambda + \mu) \bar{z} + \overline{(\lambda + \mu)} z - 2|\lambda + \mu|^2 \right\} \exp \{ 2(\bar{\lambda}\mu + \lambda\bar{\mu}) \} \\
&= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\lambda + \mu)^m \overline{(\lambda + \mu)}^n}{m!n!} J_{m,n}(z) \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{2^{i+j} (\bar{\lambda}\mu)^i (\lambda\bar{\mu})^j}{i!j!},
\end{aligned}$$

where $z, \lambda, \mu \in \mathbb{C}$. Expanding $(\lambda + \mu)^n$ and comparing coefficients immediately yield:

$$J_{a,b}(z) J_{c,d}(z) = \sum_{i=0}^{a \wedge d} \sum_{j=0}^{b \wedge c} \binom{a}{i} \binom{d}{i} \binom{b}{j} \binom{c}{j} i!j! 2^{i+j} J_{a+c-i-j, b+d-i-j}(z). \quad (3.22)$$

When $h_1 = h_2$, it follows from the relationship (2.11) that (3.21) is exact (3.22).

When $\langle h_1, h_2 \rangle_{\mathfrak{H}} = 0$, we have that

$$f \otimes_{i,j} g = \begin{cases} 0, & i + j > 0, \\ (h_1^{\otimes a} \otimes h_2^{\otimes c}) \otimes (\bar{h}_1^{\otimes b} \otimes \bar{h}_2^{\otimes d}), & i = j = 0. \end{cases}$$

Thus, (3.21) is degenerated to the relationship (2.11) in this case. \square

Remark 2. There are two another approaches to show Theorem 3.2. One is by induction over the indices c and d (see [11, Proposition 1.1.3]) using Lemma 3.1, which involves some tedious combinatorial calculations. The other is by Malliavin calculus (see [9, Theorem 2.7.10]) if we exploit the framework of complex Malliavin operators.

The following product formula which is a direct corollary of Lemma 2.3 and Theorem 3.2, will be used later.

Corollary 3.3.

$$I_{a,b}(f) \overline{I_{c,d}(g)} = \sum_{i=0}^{a \wedge c} \sum_{j=0}^{b \wedge d} \binom{a}{i} \binom{c}{i} \binom{b}{j} \binom{d}{j} i!j! I_{a+d-i-j, b+c-i-j}(f \otimes_{i,j} h),$$

where h is the reversed complex conjugate of g (see Notation 2), and

$$\begin{aligned}
& f \otimes_{i,j} h(t_1, \dots, t_{a+d-i-j}; s_1, \dots, s_{b+c-i-j}) \\
&= \int_{A^{i+j}} \mu^{i+j} (du_1 \cdots du_i dv_1 \cdots dv_j) f(t_1, \dots, t_{a-i}, u_1, \dots, u_i; s_1, \dots, s_{b-j}, v_1, \dots, v_j) \\
&\quad \times \bar{g}(s_{b-j+1}, \dots, s_{b-j+c-i}, u_1, \dots, u_i; t_{a-i+1}, \dots, t_{a-i+d-j}, v_1, \dots, v_j). \quad (3.23)
\end{aligned}$$

4 The independence of complex multiple Wiener-Itô integrals

Lemma 4.1. *For two symmetric functions $f \in \mathfrak{H}^{\odot a} \otimes \mathfrak{H}^{\odot b}$, $g \in \mathfrak{H}^{\odot c} \otimes \mathfrak{H}^{\odot d}$, let $F = I_{a,b}(f)$, $G = I_{c,d}(g)$ and h be the reversed complex conjugate of g . Then*

$$\begin{aligned} & \text{Cov}(|F|^2, |G|^2) \\ &= \sum_{i+j>0} \binom{a}{i} \binom{c}{i} \binom{b}{j} \binom{d}{j} a!b!c!d! \|f \otimes_{i,j} g\|_{\mathfrak{H}^{\otimes(m-2(i+j))}}^2 \\ &+ \sum_{r \geq 1} ((a+d-r)!(b+c-r)!)^2 \|\phi_r\|_{\mathfrak{H}^{\otimes(m-2r)}}^2, \end{aligned}$$

where

$$\phi_r = \sum_{i+j=r} \binom{a}{i} \binom{c}{i} \binom{b}{j} \binom{d}{j} i!j! f \tilde{\otimes}_{i,j} h. \quad (4.24)$$

As a consequence, the squares of the absolute values of complex multiple Wiener-Itô integrals are non-negatively correlated.

Proof. We divide the proof into three steps. Let $m = a + b + c + d$.

Firstly, it follows from Corollary 3.3, the orthogonal property and Itô's isometry of multiple Wiener-Itô integrals that

$$E[|F\bar{G}|^2] = \sum_{r \geq 0} ((a+d-r)!(b+c-r)!)^2 \|\phi_r\|_{\mathfrak{H}^{\otimes(m-2r)}}^2.$$

Secondly, we claim that

$$(a+d)!(b+c)! \|f \tilde{\otimes} h\|^2 = \sum_{i=0}^{a \wedge d} \sum_{j=0}^{b \wedge c} \binom{a}{i} \binom{d}{i} \binom{b}{j} \binom{c}{j} a!b!c!d! \|f \otimes_{i,j} g\|_{\mathfrak{H}^{\otimes(m-2(i+j))}}^2.$$

Let π (σ resp.) be a permutation of the set $\{1, \dots, a+d\}$ ($\{1, \dots, b+c\}$ resp.). Denote by π_0, σ_0 the identity permutations. We write $\pi \sim_i \pi_0$ ($\sigma \sim_i \sigma_0$ resp.) if the set $\{\pi(1), \dots, \pi(a)\} \cap \{1, \dots, a\}$ ($\{\sigma(1), \dots, \sigma(b)\} \cap \{1, \dots, b\}$ resp.) contains exactly i elements [12]. Then we have that

$$\begin{aligned} & (a+d)!(b+c)! \|f \tilde{\otimes} h\|^2 \\ &= (a+d)!(b+c)! \langle f \otimes h, f \tilde{\otimes} h \rangle \\ &= \sum_{\pi} \sum_{\sigma} \int_{A^m} d\mu^m f(t_1, \dots, t_a, s_1, \dots, s_b) \bar{g}(s_{b+1}, \dots, s_{b+c}, t_{a+1}, \dots, t_{a+d}) \\ &\quad \times \bar{f}(t_{\pi(1)}, \dots, t_{\pi(a)}, s_{\sigma(1)}, \dots, s_{\sigma(b)}) g(s_{\sigma(b+1)}, \dots, s_{\sigma(b+c)}, t_{\pi(a+1)}, \dots, t_{\pi(a+d)}) \\ &= \sum_{i=0}^{a \wedge d} \sum_{j=0}^{b \wedge c} \sum_{\pi \sim_{a-i} \pi_0} \sum_{\sigma \sim_{b-j} \sigma_0} \|f \otimes_{i,j} g\|_{\mathfrak{H}^{\otimes(m-2(i+j))}}^2 \end{aligned}$$

$$= \sum_{i=0}^{a \wedge d} \sum_{j=0}^{b \wedge c} \binom{a}{i} \binom{d}{i} \binom{b}{j} \binom{c}{j} a!b!c!d! \|f \otimes_{i,j} g\|_{\mathfrak{H}^{\otimes(m-2(i+j))}}^2.$$

Thirdly, note that when $i = j = 0$, $\|f \otimes_{i,j} g\|_{\mathfrak{H}^{\otimes(m-2(i+j))}}^2 = \|f\|^2 \|g\|^2$. Itô's isometry implies that $E[|F|^2]E[|G|^2] = a!b!c!d! \|f\|^2 \|g\|^2$. Thus,

$$\begin{aligned} & \text{Cov}(|F|^2, |G|^2) \\ &= E[|F\bar{G}|^2] - E[|F|^2]E[|G|^2] \\ &= \sum_{i+j>0} \binom{a}{i} \binom{c}{i} \binom{b}{j} \binom{d}{j} a!b!c!d! \|f \otimes_{i,j} g\|_{\mathfrak{H}^{\otimes(m-2(i+j))}}^2 \\ &+ \sum_{r \geq 1} ((a+d-r)!(b+c-r)!)^2 \|\phi_r\|_{\mathfrak{H}^{\otimes(m-2r)}}^2. \end{aligned}$$

□

Theorem 4.2. (Üstünel-Zakai independence criterion) *For two symmetric functions $f \in \mathfrak{H}^{\odot a} \otimes \mathfrak{H}^{\odot b}$, $g \in \mathfrak{H}^{\odot c} \otimes \mathfrak{H}^{\odot d}$ with $a + b \geq 1$, $c + d \geq 1$, the following conditions are equivalent:*

- (i) $I_{a,b}(f)$ and $I_{c,d}(g)$ are independent random variables;
- (ii) $f \otimes_{1,0} g = 0$, $f \otimes_{0,1} g = 0$, $f \otimes_{1,0} h = 0$ and $f \otimes_{0,1} h = 0$ a.e. μ^{m-2} , where $m = a + b + c + d$ and h is the reversed complex conjugate of g .

Proof. (i) \Rightarrow (ii): Denote $F = I_{a,b}(f)$, $G = I_{c,d}(g)$. It follows from Proposition 2.4 (1) that inside a fixed Wiener chaos (i.e., for the fixed (a, b)), all the L^q -norms ($q > 1$) are equivalent. Thus $\text{Cov}(|F|^2, |G|^2)$ is finite. If (i) is satisfied then $\text{Cov}(|F|^2, |G|^2) = 0$. It follows from Lemma 4.1 that $f \otimes_{1,0} g = 0$ and $f \otimes_{0,1} g = 0$. Note that $\bar{G} = I_{d,c}(h)$ and F, \bar{G} are also independent random variables. Thus we also have that $f \otimes_{1,0} h = 0$ and $f \otimes_{0,1} h = 0$.

(ii) \Rightarrow (i): We divide the proof into three steps along the same line as the proof for real multiple integrals [8].

Firstly, let $\mathcal{H}_f, \mathcal{G}_f$ respectively denote the Hilbert subspace in \mathfrak{H} spanned by all functions

$$\begin{aligned} t &\mapsto \int_{A^{a+b-1}} f(t, x_1, \dots, x_{a+b-1}) h(x_1, \dots, x_{a+b-1}) \mu^{a+b-1}(dx_1 \dots dx_{a+b-1}) \\ t &\mapsto \int_{A^{a+b-1}} f(x_1, \dots, x_{a+b-1}, t) h(x_1, \dots, x_{a+b-1}) \mu^{a+b-1}(dx_1 \dots dx_{a+b-1}) \end{aligned}$$

where $t \in A$ and $h \in \mathfrak{H}^{\otimes(a+b-1)}$. Similarly we define $\mathcal{H}_g, \mathcal{G}_g$ in terms of g . Denote by $\overline{\mathcal{G}_f}$ the complex conjugate of \mathcal{G}_f . We claim that condition (ii) implies that $\{\mathcal{H}_f, \overline{\mathcal{G}_f}\}$ and $\{\mathcal{H}_g, \overline{\mathcal{G}_g}\}$ are orthogonal. In fact, $f \otimes_{1,0} g = 0$, $f \otimes_{0,1} g = 0$, $f \otimes_{1,0} h = 0$ and $f \otimes_{0,1} h = 0$ respectively imply that $\mathcal{H}_f \perp \overline{\mathcal{G}_g}$, $\overline{\mathcal{G}_f} \perp \mathcal{H}_g$, $\mathcal{H}_f \perp \mathcal{H}_g$ and $\overline{\mathcal{G}_f} \perp \overline{\mathcal{G}_g}$.

using Fubini Theorem. For example, let $h(x) \in \mathfrak{H}^{\otimes(a+b-1)}$, $l(y) \in \mathfrak{H}^{\otimes(c+d-1)}$ and $m = a + b + c + d$, we have that

$$\begin{aligned} & \int_A \mu(dt) \int_{A^{a+b-1}} f(t, x) h(x) \mu^{a+b-1}(dx) \int_{A^{c+d-1}} g(y, t) l(y) \mu^{c+d-1}(dy) \\ &= \int_{A^{m-2}} \mu^{m-2}(dxdy) h(x) l(y) \int_A f(t, x) g(y, t) \mu(dt) \\ &= \int_{A^{m-2}} h(x) l(y) f \otimes_{1,0} g \mu^{m-2}(dxdy) = 0 \end{aligned}$$

i.e., \mathcal{H}_f and $\overline{\mathcal{G}_g}$ are orthogonal.

Secondly, let $\{\varphi_n\}$ ($\{\psi_n\}$ resp.) be an orthonormal basis for the Hilbert subspace in \mathfrak{H} spanned by $\{\mathcal{H}_f, \overline{\mathcal{G}_f}\}$ ($\{\mathcal{H}_g, \overline{\mathcal{G}_g}\}$ resp.). Since the tensor products $\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_a} \otimes \bar{\varphi}_{j_1} \otimes \cdots \otimes \bar{\varphi}_{j_b}$ form an orthonormal basis in $\mathcal{H}^{\otimes a} \otimes \overline{\mathcal{G}_f}^{\otimes b}$, it follows from monotonic class theorem [4, 5] that f (and g) can be decomposed as

$$\begin{aligned} f &= \sum e_{i_1 \dots i_a j_1 \dots j_b} \varphi_{i_1} \otimes \cdots \otimes \varphi_{i_a} \otimes \bar{\varphi}_{j_1} \otimes \cdots \otimes \bar{\varphi}_{j_b}, \\ g &= \sum l_{i_1 \dots i_c j_1 \dots j_d} \psi_{i_1} \otimes \cdots \otimes \psi_{i_c} \otimes \bar{\psi}_{j_1} \otimes \cdots \otimes \bar{\psi}_{j_d}. \end{aligned}$$

Thirdly, we claim that F, G are independent. In fact, we write $\xi_i = \int_A \varphi_i(a) M(da)$ and $\eta_j = \int_A \psi_j M(da)$ for all i and j . Then the Cramér-Wold theorem implies that the entire sequences $\{\xi_i\}$ and $\{\eta_j\}$ are independent [8]. It follows from (2.11) that $F = I_{a,b}(f)$ ($G = I_{c,d}(g)$ resp.) can be expanded into polynomials in ξ_1, ξ_2, \dots (η_1, η_2, \dots resp.). Thus F, G are independent. \square

Remark 3. Similar to real multiple integrals [15], condition (i) of Theorem 4.2 is also equivalent to

$$(iii) \quad \text{Cov}(|F|^2, |G|^2) = 0, \text{ i.e., } |F|^2, |G|^2 \text{ are uncorrected,}$$

which can be observed from the above proof.

5 Appendix: Asymptotic independence of complex multiple Wiener-Itô integrals

Definition 5.1. [10, Definition 3.3] Fix $d \geq 1$ and for each $n \geq 1$, let $F_n = (F_{1,n}, \dots, F_{d,n})$ be a d -dimensional complex-valued random variable. We say the variables $(F_{i,n})_{1 \leq i \leq d}$ are asymptotically moment-independent if each $F_{i,n}$ admits moments of all orders and for any two sequences (l_1, \dots, l_d) and (k_1, \dots, k_d) of non-negative integers,

$$\lim_{n \rightarrow \infty} \left\{ E \left[\prod_{i=1}^d F_{i,n}^{l_i} \bar{F}_{i,n}^{k_i} \right] - \prod_{i=1}^d E[F_{i,n}^{l_i} \bar{F}_{i,n}^{k_i}] \right\} = 0. \quad (5.25)$$

Theorem 5.2. (Nourdin-Rosiński asymptotically moment-independence criterion) Fix $d \geq 2$ and let (a_1, \dots, a_d) and (b_1, \dots, b_d) be two sequences of non-negative integers. For each $n \geq 1$, let $F_n = (F_{1,n}, \dots, F_{d,n})$ be a d -dimensional complex multiple Wiener-Itô integrals, where $F_{i,n} = I_{a_i, b_i}(f_{i,n})$ with $f_{i,n} \in \mathfrak{H}^{\odot a_i} \otimes \mathfrak{H}^{\odot b_i}$. If for every $1 \leq i \leq d$,

$$\sup_n E[|F_{i,n}|^2] < \infty, \quad (5.26)$$

then the following conditions are equivalent:

- (i) the random variables $(F_{i,n})_{1 \leq i \leq d}$ are asymptotically moment-independent;
- (ii) $\lim_{n \rightarrow \infty} \text{Cov}(|F_{i,n}|^2, |F_{j,n}|^2) = 0$ for every $i \neq j$;
- (iii) For every $i \neq j$, $\lim_{n \rightarrow \infty} \|f_{i,n} \otimes_{r,s} f_{j,n}\| = 0$ for every (r,s) such that $r \leq a_i \wedge b_j$, $s \leq a_j \wedge b_i$, $r + s > 0$, and $\lim_{n \rightarrow \infty} \|f_{i,n} \otimes_{r,s} h_{j,n}\| = 0$ for every (r,s) such that $r \leq a_i \wedge a_j$, $s \leq b_i \wedge b_j$, $r + s > 0$, where $h_{j,n}$ is the reversed complex conjugate of $f_{j,n}$.

Proof. Suppose that $F_{i,n} = U_{i,n} + \sqrt{-1}V_{i,n}$. Thus, it follows from Theorem 3.4 and Remark 3.5 of [10] and Proposition 2.5 that the random vectors $(U_{i,n}, V_{i,n})$, $i = 1, \dots, d$ being asymptotically moment-independent, i.e., for any sequence (l_1, \dots, l_d) and (k_1, \dots, k_d) ,

$$\lim_{n \rightarrow \infty} \left\{ E\left[\prod_{i=1}^d U_{i,n}^{l_i} V_{i,n}^{k_i}\right] - \prod_{i=1}^d E[U_{i,n}^{l_i} V_{i,n}^{k_i}] \right\} = 0 \quad (5.27)$$

is equivalent to that $\lim_{n \rightarrow \infty} \text{Cov}(|F_{i,n}|^2, |F_{j,n}|^2) = 0$ for every $i \neq j$. It is easy to check that (5.25) is equivalent to (5.27). Thus, (i) is equivalent to (ii).

Using (2.3), it follows from Lemma 4.1 that (ii) is equivalent to (iii). \square

Corollary 5.3. (Nourdin-Rosiński joint convergence criterion) Under notation of Theorem 5.2, let $(\eta_i)_{1 \leq i \leq d}$ be a complex random vector such that

- (i) As $n \rightarrow \infty$, $F_{i,n}$ converges in law to η_i for each $1 \leq i \leq d$;
- (ii) The random variables η_1, \dots, η_d are independent;
- (iii) Condition (ii) or (iii) of Theorem 5.2 holds;
- (iv) The law of η_i is determined by its moments for each $1 \leq i \leq d$.

Then the joint convergence holds, i.e.,

$$(F_{1,n}, \dots, F_{d,n}) \xrightarrow{\text{law}} (\eta_1, \dots, \eta_d), \quad \text{as } n \rightarrow \infty. \quad (5.28)$$

Remark 4. In the above condition (iv), we do not assume that the laws of both the real part and the imaginary part of η_i are determined by their moments. This is the difference between Corollary 3.6 of [10] and Corollary 5.3.

Proof. Note that Theorem 3 of [13] still holds for probability measures on \mathbb{C}^d . Precisely stated, if each of d coordinate projections $P_i(\mu)$ of a probability measure μ on \mathbb{C}^d is uniquely determined by its moments sequence, then the measure μ is also uniquely determined by its moments. By applying Proposition 2.4 (2), we can show the desired conclusion by means of modifying the proof of Corollary 3.6 of [10] slightly. \square

In analogy with the characterization of moment-independence of limits of real multiple Wiener-Itô integrals [10, Theorem 3.1], we can deduce the following corollary for complex multiple Wiener-Itô integrals from the above proof.

Corollary 5.4. (Nourdin-Rosiński moment-independence criterion of limits) *Under notations of Theorem 5.2 and Corollary 5.3. Assume that (5.28) holds. Then η_i 's admit moments of all orders and the following conditions are equivalent:*

(α) *The random variables $(\eta_i)_{1 \leq i \leq d}$ are moment-independent, i.e.,*

$$E\left[\prod_{i=1}^d \eta_i^{l_i} \bar{\eta}_i^{k_i}\right] = \prod_{i=1}^d E[\eta_i^{l_i} \bar{\eta}_i^{k_i}], \quad \forall k_1, \dots, k_d, l_1, \dots, l_d \in \mathbb{N};$$

(β) $\lim_{n \rightarrow \infty} \text{Cov}(|F_{i,n}|^2, |F_{j,n}|^2) = 0$ *for every $i \neq j$.*

Moreover, if condition (iv) of Corollary 5.3 is satisfied, then (α) is equivalent to that

(γ) *The random variables $(\eta_i)_{1 \leq i \leq d}$ are independent.*

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